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LETTER TO THE EDITOR

On the relationship between the critical exponents of percolation conductivity and static exponents of percolation

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Abstract. We argue that the critical exponent t of random conductance networks near the percolation threshold is given by $t=(d-1)\nu$ for low dimensionalities and $t=1+\beta'$ for high dimensionalities, where ν is the correlation length exponent, β' the backbone exponent and d is dimensionality. We argue that what separates the two regimes is a critical fractal dimensionality D_l which equals 2. We also argue that D_l is also a critical fractal dimensionality for fractals such as lattice animals and diffusion-limited aggregates. The result for low dimensionalities has been also obtained by Aharony and Stauffer by a different argument.

Random conductance networks have become an important model for the investigation of transport processes and other phenomena in disordered systems. Much of the current interest in the properties of such networks is focused on the vicinity of the percolation threshold p_c. As the fraction p of conducting bonds of a network in which the rest of the bonds are insulators approaches p_c from above, the DC conductivity *vanishes as*

$$\Sigma \sim (p-p_c)^t \tag{1}$$

whereas the conductivity of a network in which a fraction p of bonds are superconductors and the rest ordinary conductors *diverges* as p_c is approached from below,

$$\Sigma \sim (p_c-p)^{-s}. \tag{2}$$

The exponents t and s are believed to be universal and independent of each other. Moreover, in two dimensions a duality argument (Dykhne 1970, Straley 1977) establishes that $s=t$. t is a dynamical exponent whereas s is a static one.

In the past several years many authors have attempted to obtain accurate estimates for exponents t and s. A list of these authors is too long to be given here, but in figure 1 we present the frequency distributions of the reported values of t in two and three dimensions. The distribution of values of t at $d=2$ represents 27 data points, while that of t at $d=3$ consists of 20 data points. An outstanding problem is whether there exist simple relations between t and s and the static exponents of percolation. Since conduction takes place only on the backbone of the infinite cluster, the problem is very difficult to solve unless one makes specific assumptions about the structure of the backbone. From their hypothesis about the density of states on the largest percolation

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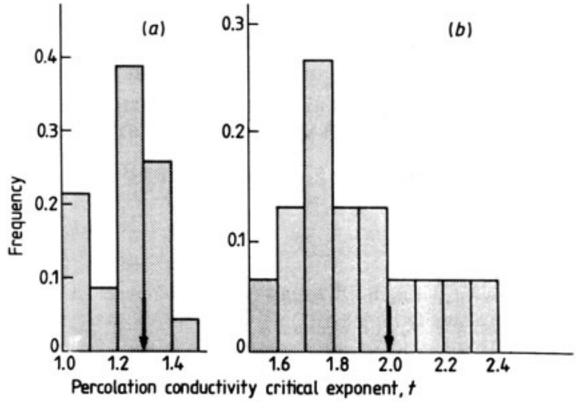

Figure 1. The frequency distribution of the reported values of percolation conductivity exponent t_c. Arrows indicate the currently accepted values. (a) Two-dimensional values, (b) three-dimensional values.

cluster at p_c and based on numerical evidence, Alexander and Orbach (1982) conjectured that

$$t = \tfrac{1}{2}[(3d-4)\nu - \beta], \tag{3}$$

where ν is the critical exponent of percolation correlation length, β the critical exponent of $P_\infty(p)$, the strength of the infinite network, and d is dimensionality.

In this letter we discuss the applicability of the Alexander–Orbach (AO) conjecture and propose alternative relations. To begin with, we note that an ε expansion ($\varepsilon = 6-d$) has been developed for the exponent t by Harris *et al* (1984). Their result is a corrected version of the earlier works of Stephen (1978) and Dasgupta *et al* (1978). These authors used the scaling relation $t=(d-2)\nu+\zeta$ to obtain the ε expansion for t. This is similar to the predicted t if one accepts the so-called 'nodes and links' model of the backbone first proposed by Skal and Shklovskii (1974) and de Gennes (1976). In this model the backbone is assumed to be made of nodes which are connected by long chains of many bonds called links. The length of a link is of the order of the percolation correlation length ξ_p and ζ represents the critical exponent which describes the divergence of the resistance of the links. The work of Harris *et al* (1984) indicates that to linear order in ε one has $\zeta = 1+\varepsilon/42$, in contrast with the earlier claim of Wallace and Young (1978) and Stephen (1978) that $\zeta = 1$ at *all* dimensions. Thus with $\nu = \frac{1}{2} + 5\varepsilon/84$ one obtains $t = 3 - 10\varepsilon/42$, whereas the AO conjecture yields $t = 3 - 11\varepsilon/42$ if one uses $\beta = 1-\varepsilon/7$. Based on this discrepancy Harris *et al* (1984) have argued that the AO conjecture is not exact. We believe that the question of ε expansion for the conduction exponents remains open until one develops a technique by which ε expansions for t and s can be determined *without resorting to any specific scaling law.* It is only then that the ε expansions can be used to test the validity of a conjecture.

Very recent and accurate estimation of t and s by Zabolitzky (1984), Lobb and Frank (1984), Hong *et al* (1984) and Herrmann *et al* (1984) indicate that the AO conjecture may fail at $d=2$. These authors have obtained $t(d=2)=s(d=2)\simeq 1.30$, in contrast with the AO conjecture, $t=\frac{91}{72}\simeq 1.26388$. Coniglio and Stanley (1984) presented an argument according to which $s=\nu(d/D+D-d)$, for $D\leqslant 2$ where $D=d-\beta/\nu$ is the fractal dimension of the largest percolation cluster at p_c. This yields $s(d=2)=t(d=2)=\frac{16612}{13104}\simeq 1.26770$, in disagreement with the estimate of $t(d=2)$.

We now present alternative relations which relate t to the static exponents of percolation. From the similarity between Kirchoff's equations and the equations describing spin waves at low temperatures, one can establish an exact correspondence between classical Heisenberg ferromagnets and random resistor networks (Kirkpatrick 1973, Stinchcombe 1979). Halperin and Hohenberg (1969) have derived, within a hydrodynamic theory, an expression for the Fourier transform of the Green function of the transverse excitations of a Heisenberg ferromagnet at low temperatures. This Green function is equivalent to the Green function G_{ij} for the DC conduction problem. For the DC conduction problem G_{ij} represents a response function because it represents the response (the voltage) at site j if a unit current is injected at site i. Therefore, if one employs the correspondence between Heisenberg ferromagnets and random resistor networks, the Halperin–Hohenberg expression can be translated into an equation for $\hat{G}_{ij}(\boldsymbol{k})$, the Fourier transform of G_{ij}. This equation in the limits $k \to 0$ and $k\xi_p \ll 1$ is given by

$$\hat{G}_{ij}(\boldsymbol{k}) = P_\infty^2(p)/[\Sigma(p)|\boldsymbol{k}^2|F(\xi_p(p)\boldsymbol{k})]. \tag{4}$$

Here F is a scaling function which remains regular in the limit $k \to 0$. The governing equation for G_{ij} is (Blackman 1976, Sahimi *et al* 1983) (see also Lubensky 1977)

$$\sigma_m\left(ZG_{ij} - \sum_l G_{lj}\right) = -\delta_{ij}, \tag{5}$$

where Z is the coordination number of the network and the sum is over all nearest neighbours of j. σ_m has the unit of conductivity and its significance is discussed below. If we take the Fourier transform of (5) and expand the trigonometric functions that arise in Fourier transforming, we find, in the limit $k \to 0$, that

$$\hat{G}_{ij}(\boldsymbol{k}) \sim 1/|\boldsymbol{k}^2|\sigma_m, \tag{6}$$

where we have restricted our attention to a simple cubic lattice in d dimensions. Substitution of (6) into (4) yields

$$\Sigma(p) \sim \sigma_m P_\infty^2(p). \tag{7}$$

In the Green function formulation of conduction (Blackman 1976) and diffusion (Sahimi *et al* 1983) σ_m is taken as an approximation to the true conductivity (diffusivity) of the network which can be calculated by an effective medium approximation (EMA), in which case $\sigma_m \sim (p - p_c)$. This means that

$$t = 1 + 2\beta. \tag{8}$$

Essam *et al* (1974) presented a modified EMA for the Bethe lattice which was capable of producing the exact result $t_m = 2$ for the *microscopic* conductivity of the lattice. Hughes and Sahimi (1983, unpublished) noted that this modified EMA can be refined one more step so that it can produce the exact result $t = 3$ for the *macroscopic* conductivity of the Bethe lattice. In this case an equation similar to (7) is produced. Hughes and Sahimi (see Sahimi 1983) hypothesised that an equation similar to (7) may hold at *all* dimensions, thus conjecturing (8). More recently, Kholodenko and Freed (1984) produced (8) by estimating in two different ways the diffusion coefficient for random walks on percolating networks as $p \to p_c$ from above. Equation (8) predicts that $t(d=2) = \frac{92}{72} \simeq 1.2777$. This currently represents the most accurate theoretical prediction of t (and s) at $d = 2$ as compared with the most recent estimates mentioned above. This is not really surprising as the EMA is very accurate at low dimensions.

Note that (8) yields $t = 3 - 12\varepsilon/42$. However, (8) appears to underestimate t at higher dimensions as is evident from table 1 where we compare the predictions of (8) with the available data and with the AO conjecture, although the predictions of (8) would be consistent with the data if the statistical uncertainty of the data is taken into account.

Table 1. Comparison of the available data for t with the predictions of the AO conjecture and those of equations (8) and (13).

d	β	β'	ν	t (data)	t (AO)	t (equation (8))	t (equation (13))
2	5/36	0.4[a]	4/3	1.297[b]	$91/72 \simeq 1.2638$	$92/72 \simeq 1.278$	—
3	0.42[c]	1.03[d]	0.88[e]	2.02[f]	1.99	1.84	2.03
4	0.62[g]	1.39[d]	0.66[g]	2.39[f]	2.33	2.24	2.39
5	0.84[g]	1.74[d]	0.57[g]	2.73[f]	2.71	2.68	2.74
$\geqslant 6$	1	2[h]	$\frac{1}{2}$	3	3	3	3

[a] Li and Strieder (1982);
[b] Zabolitzky (1984), Lobb and Frank (1984), Hong *et al* (1984), Herrmann *et al* (1984);
[c] Margolina *et al* (1982);
[d] Hong and Stanley (1983);
[e] Heermann and Stauffer (1981), Gaunt and Sykes (1983);
[f] Adler (1984) based on reanalysis of the data of Fisch and Harris (1978) (Pandey and Stauffer (1983) give $t \simeq 2.0$);
[g] Fisch and Harris (1978);
[h] Larson and Davis (1982).

According to (6), $1/\sigma_m$ represents the resistance between two widely separated sites i and j. Thus if $\sigma_m \sim (p - p_c)^{\zeta_R}$, one obtains

$$t = \zeta_R + 2\beta, \tag{9}$$

where the exponent ζ_R is expected to be universal and $\zeta_R \neq \zeta$. On a Bethe lattice the resistance between two sites separated by a distance ξ_p is proportional to ξ_p^2; thus $\zeta_R = 1$. This is also true for six- and higher-dimensional systems for which the nodes and links model is exact. This simple model breaks down below six dimensions and ζ_R is a dimensional dependent quantity. A glance at table 1 reveals that the dimensional dependence of ζ_R is not monotonic.

It remains only to relate ζ_R to other percolation exponents. We believe that there are two separate regimes in each of which ζ_R is related to other percolation exponents by a different relation. At low dimensions two widely separated nodes are connected by complicated parallel paths. At higher dimensions the effect of parallel paths decreases and the nodes and links picture becomes increasingly more accurate. Thus there should be a lower critical dimensionality d_l which separates the two regimes. We note here that Fucito and Parisi (1981) have shown that the ε expansions for the critical exponents calculated from a φ^3 theory break down at some anomalous dimension. At this anomalous dimension the fourth-order potential becomes relevant. They estimated that this anomalous dimension lies between two and three. A similar phenomenon was observed by Harris (1983) and Harris and Lubensky (1983) in their field-theoretic formulation of the backbone problem. Coniglio and Stanley (1984) have also argued that there are two distinct scaling relations that relate the exponent s to other percolation exponents. They argued that there is a lower critical fractal dimensionality D_l which separates the two regimes and that $D_l = 2$. A similar argument was made by Aharony and Stauffer (1984) who argued that the AO conjecture may break down below $D_l = 2$ and presented an alternative relation relating t to other percolation exponents for $D \leqslant D_l$. This lower critical fractal dimensionality corresponds to a value of $d_l \simeq 2.2$, in agreement with the calculations of Fucito and Parisi (1981). We conjecture that the relevance of the fourth-order potential may also be responsible for the breakdown of a scaling law which may relate t or s to other percolation exponents.

It remains to relate ζ_R to other percolation exponents for $d \leqslant d_l$ and $d \geqslant d_l$. It has been recently argued that (Sahimi 1984)

$$s = \nu, \qquad d \leqslant d_l, \tag{10}$$

which is different from the relation proposed by Coniglio and Stanley (1984) mentioned above. This equation satisfies the exact result $s(d=1) = 1$ and it also satisfies the $1 + \varepsilon_1$ expansion, where $\varepsilon_1 = d - 1$; it yields $s/\nu = 1$, in agreement with Kirkpatrick (1977). Since $s(d=2) = t(d=2)$, one obtaind $\zeta_R(d=2) = \nu(d=2) - 2\beta(d=2) = \gamma(d=2) - \nu(d=2)$, where γ is the susceptibility exponent. If we assume that this last relation holds for all d such that $d \leqslant d_l$, i.e. $\zeta_R = \gamma - \nu$ for $d \leqslant d_l$, we obtain

$$t = (d-1)\nu, \qquad d \leqslant d_l. \tag{11}$$

This is the same as the result of Aharony and Stauffer (1984). They obtained (11) by a completely different argument. Equation (11) predicts that $t(d=1) = 0$, an exact (and trivial) result, and $t(d=2) = \nu$. The accuracy of the result $t(d=2) = \nu$ is presently slightly less than $t(d=2) = 1 + 2\beta(d=2)$, as compared with the available data. However, (11) also satisfies the $1 + \varepsilon_1$ expansion as pointed out by Aharony and Stauffer (1984). Aharony and Stauffer (1984) also argued that one probably needs to simulate very large systems to observe the result $t(d=2) = \nu$. Equation (11) had earlier been conjectured by Levinshtein *et al* (1975) to hold at *all* dimensions. However, (11) is definitely wrong for $d \geqslant 6$, because it yields $t = \frac{5}{2}$, whereas $t(d \geqslant 6) = 3$. The result $t(d=2) = s(d=2) = \nu$ had been conjectured by Straley (1980).

Several points deserve notice. Equations (10) and (11) together satisfy Straley's (1980) hyperscaling law, $t + s = \nu d$. Straley (1980) had conjectured that this law holds at all d. Our results indicate that it should hold at least for $d \leqslant d_l$. We also note that the result $\zeta_R = \gamma - \nu$ satisfies the inequalities $0 \leqslant \zeta_R \leqslant \nu$, for $1 \leqslant d \leqslant d_l$ (Skal and Shklovskii 1974). This result also shows that the resistive susceptibility exponent γ_r defined by Fisch and Harris (1978) is given by $\gamma_r = \gamma + \nu$, for $d \leqslant d_l$. In two dimensions this yields $\gamma_r = \frac{67}{18} \simeq 3.72$, in good agreement with the recent result of Adler (1984) who reanalysed the series expansion data of Fisch and Harris (1978) to take into account the effect of non-analytic confluent corrections and obtained $\gamma_r \simeq 3.73$. We finally note that if one takes (11) seriously, one obtains $d_s = 2D/(1+D)$, where d_s is the spectral dimension of the largest percolation cluster at p_c. If one considers a fractal of linear dimension L, the *conductance* g of the fractal scales as $g \sim L^{\beta_L}$, where β_L was argued by Rammal and Toulouse (1983) to be $\beta_L = D(1 - 2/d_s)$. This means that for the largest percolation cluster at p_c and for $d \leqslant d_l$, or equivalently $D \leqslant D_l$, one has

$$\beta_L = -1, \qquad d \leqslant d_l, \tag{12}$$

i.e. β_L is *independent of dimension* for $1 \leqslant d \leqslant d_l$. These results have several other implications for lattice animals and diffusion-limited aggregates and also for the work of Coniglio and Stanley (1984). We conjecture that $D_l = 2$ is also a lower critical fractal dimensionality for *all* fractals which have homogeneous interior structure, such as lattice animals. Thus a hypothesis such as the AO conjecture may hold for such fractals with $D > D_l$, but not for $D \leqslant D_l$. These matters will be discussed elsewhere.

For $d \geqslant d_l$, we have not been able to relate ζ_R to other percolation exponents. It is easily possible to relate ζ_R to the various susceptibility exponents defined by Fisch and Harris (1978). However, our aim is a scaling law which relates directly ζ_R to other well studied percolation exponents such as ν, β and γ. We have noticed that the scaling law

$$t = 1 + \beta', \qquad d \geqslant d_l, \tag{13}$$

where β' is the critical exponent of the backbone yields excellent predictions of t for systems above d_l. This is evident from table 1 where we also list the predictions of (13). This equation appears to fail at $d = 2$, if the current value $\beta'(d = 2) \simeq 0.4$ is accepted as an accurate estimate. Harris and Lubensky (1983) and Harris (1983) have shown that

$$\beta' = 2\beta + \nu\psi, \tag{14}$$

where ψ is a new and independent crossover exponent which describes the correlation function of the backbone. Substitution of (14) into (13) yields

$$t = 1 + 2\beta + \nu\psi, \tag{15}$$

which means that $\zeta_R = 1 + \nu\psi$. The work of Fortuin and Kasteleyn (1972) shows that the resistance between two widely separated sites in a random resistor network plays a role similar to the thermal correlation function in the equivalent q-state Potts model in the limit $q \to 0$. From this point of view it is very satisfying that ζ_R and ψ seem to be related. It is also satisfying that ζ_R seems to be related to an exponent describing a backbone property, since conduction takes place only on the backbone. The results of Harris (1983) and Harris and Lubensky (1983) also indicate that ψ has non-monotonic dependence on d which is consistent with our result. One may also interpret the $\nu\psi$ term as the correction to the EMA approximation which yielded (8). If we assume that (13) holds for all $d > 1$, then in view of (10) and (11) one must have $\beta'(d = 2) = \nu(d = 2) - 1 = \frac{1}{3}$. This is close to the current value of β' mentioned above, but appears to be somewhat low.

Harris (1983) and Harris and Lubensky (1983), who formulated the field-theoretic approach to the backbone problem, obtained the result $\psi = 2\varepsilon^2/49$. They also pointed out that this field-theoretic formulation of the backbone problem should break down at some low dimension d^*, where the fixed point of φ^3 theory becomes unstable with respect to a φ^4 interaction and thus a φ^4 perturbation is relevant. This presumably means that (14) breaks down below d^*. We believe that $d^* = d_l \simeq 2.2$. If one believes that t and s vary continuously with d, then (3), (11) and (13) should yield the same t at $d_l \simeq 2.2$. We have checked this and it turns out that the AO conjecture yields $t(d = d_l) \simeq 1.51$, equation (11) yields $t(d = d_l) \simeq 1.48$ and (13) yields $t(d = d_l) \simeq 1.52$. Thus, taking into account the statistical uncertainty of the available data, these equations seem to be consistent with each other. On the other hand (8) yields $t(d = d_l) \simeq 1.39$. Future works, both theoretical and numerical, will further assess the validity of (3), (11) and (13).

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